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ON POSITIONAL CONTROL UNDER AFTEREFFECT IN THE CONTROLLING FORCES*

Iu.S. OSIPOV and V.G. PIMENOV

Positional control problems are studied for systems with aftereffect in the controls. The existence of an equilibrium situation is proved in an encounter-evasion problem and a method is indicated for constructing the desired controls. The article abuts the investigations in /1—5/ and is a continuation of /6/.

1. The controlled system

$$x^{\cdot}(t) = f_1(t, x(t), u(t), u(t-\tau)) + f_2(t, x(t), v(t)) \tag{1.1}$$
$$x \in R^n,\ u \in P \subset R^{r_1},\ v \in Q \subset R^{r_2},$$
$$t \in [t_0, \vartheta],\ \tau = \text{const} > 0$$

is specified. Here x is the phase vector, u and v are controls, P and Q are compacta (R^n is an n-dimensional Euclidean space), the functions $f_1(t, x, u, w)$ and $f_2(t, x, v)$ are defined, are continuous in all arguments, and satisfy a local Lipschitz condition in x on $[t_0, \vartheta] \times R^n \times$ and $[t_0, \vartheta] \times R^n \times Q$, respectively, and in their domains

$$\| f_1(t, x, u, w) + f_2(t, x, v)\| \leqslant \varkappa(1 + \|x\|),\ \varkappa = \text{const}$$

Let U_* be some set of functions on the interval $[-\tau, 0]$ and $M_* \subset R^n$. Two problems are examined. The first consists in the construction of a control u by the feedback principle $u[t] = u(t; x[t], u[t+s], -\tau \leqslant s < 0)$, taking the vector x onto M_* at some instant $t_* \leqslant \vartheta$ for any admissible realization of control v, and in such a way that the condition $u[t_* + s] \in U_*$ is fulfilled (this is an encounter problem /6/). The second problem is to construct a control v by the feedback principle $v[t] = v(t; x[t]; u[t+s], -\tau \leqslant s < 0)$, guaranteeing that the phase vector of system (1.1) evades contact with M_* for any admissible realization of control u (an evasion problem). Certain variants of these problems were studied in /6/ from the viewpoint of differential game theory developed in /1—3/. In the present paper, in constrast to /6/, we examine the general situation, obtain the necessary and sufficient conditions for the solvability of the problems posed, and indicate a method for constructing the optimal controls.

Let us pose the problems more precisely. Let $P(\sigma)$ be a collection of all measurable functions $u(\cdot)$ on set σ with values in P, $Q(\sigma)$ be a collection of all measurable functions $v(\cdot)$ on set σ with values in Q. Every triple $p = \{t; x; u(s), -\tau \leqslant s < 0\}$, where $t \in [t_0, \vartheta]$, $x \in R^n$, $u(\cdot) \in P([-\tau, 0))$, is called a position. A rule that associates a function $u(t)$ from $P([t_*, t^*))$ ($v(t)$ from $Q([t_*, t^*))$ with a position $p_* = \{t_*, x_*, u_*(s)\}$ and a number $t^* \in (t_*, \vartheta]$, is called a strategy $U(V)$. Let there be specified a position $p_0 = \{t_0, x_0, u_0(s)\}$ and a partitioning Δ of interval $[t_0, \vartheta]$ by the points $\tau_0 = t_0 < \tau_1 < \ldots < \tau_{N(\Delta)} = \vartheta$, $\delta(\Delta) = \max_i(\tau_{i+1} - \tau_i)$. We define an approximate motion of system (1.1), corresponding to strategy U, as the pair $\{x[\cdot]_\Delta, u[\cdot]_\Delta\}$, where the absolutely continuous function $x[\cdot]_\Delta = x[t]_\Delta = x[t, p_0, U]_\Delta$, $t_0 \leqslant t \leqslant \vartheta$, and the control $u[\cdot]_\Delta \in P([t_0 - \tau, \vartheta])$ satisfy the conditions

$$x[t_0]_\Delta = x_0,\ u[t_0 + s]_\Delta = u_0(s),\ -\tau \leqslant s < 0 \tag{1.2}$$

in addition, the equality

$$x^{\cdot}[t]_\Delta = f_1(t, x[t]_\Delta, u[t]_\Delta, u[t-\tau]_\Delta) + f_2(t, x[t]_\Delta, v[t]) \tag{1.3}$$

is fulfilled for almost all $t \in [t_0, \vartheta]$ and for $t \in [\tau_i, \tau_{i+1})$ $u[t]_\Delta$ is a function from $P([\tau_i, \tau_{i+1}))$, designated as the strategy U with respect to position $\{\tau_i, x[\tau_i]_\Delta, u_{\tau_i}[s]_\Delta\}$ and to number τ_{i+1}. Here and subsequently, $u_t(s) = u(t+s)$, $-\tau \leqslant s < 0$. In (1.3) $v[t] \in Q([t_0, \vartheta])$ is some realization of control v. We define an approximate motion of system (1.1), corresponding to strategy V, as the pair $\{x[\cdot]_\Delta, u[\cdot]\}$, where the absolutely continuous function $x[\cdot]_\Delta = x[t]_\Delta = x[t, p_0, V]_\Delta$, $t_0 \leqslant t \leqslant \vartheta$, satisfies condition (1.2) and for almost all $t \in [t_0, \vartheta]$ satisfies the equations

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$$x^{\cdot}[t]_\Delta = f_1(t, x[t]_\Delta, u[t], u[t-\tau]) + f_2(t, x[t]_\Delta, v[t]_\Delta)$$

where for $t \in [\tau_i, \tau_{i+1}), v[t]_\Delta$ is a function from $Q([\tau_i, \tau_{i+1}))$, designated as the strategy V with respect to position $\{\tau_i, x[\tau_i]_\Delta, u_{\tau_i}[s]\}$ and to number τ_{i+1}. The function $u[\cdot] = u[t] \in P([t_0 - \tau, \vartheta])$ is some realization of control u, satisfying the condition $u[t_0 + s] = u_0(s), -\tau \leqslant s < 0$.

Let $[A]$ be the closure of $A \subset R^n$ and A^ε be an open ε-neighborhood of A. Let some set M be prescribed in the position space. By M_t we denote the section of M by t (i.e., the set of pairs $\{x, u(s)\}$ such that $\{t, x, u(s)\} \in M$; by $M_{t, u(s)}$ we denote the sections by t and $u(s)$. By $[M]$ and M^ε we denote collections of positions $\{t, x, u(s)\}$ such that $x \in [M_{t, u(s)}]$ and $x \in M^\varepsilon_{t, u(s)}$, respectively.

Problem 1 (encounter). System (1.1), position p_0 and set M are prescribed. Construct a strategy U° with the property: for any $\varepsilon > 0$ we can find $\delta_0 > 0$ such that the condition

$$x[\eta]_\Delta \in M^\varepsilon_{\eta, u^\circ_\eta[s]_\Delta}$$

is fulfilled at some instant $\eta \in [t_0, \vartheta]$ for every motion $\{x[t]_\Delta, u^\circ[\cdot]_\Delta\} = \{x[t, p_0, U]^\circ_\Delta, u^\circ[\cdot]_\Delta\}$ with $\delta(\Delta) \leqslant \delta_0$.

Problem 2 (evasion). System (1.1), position p_0 and set M are prescribed. Construct a strategy V° with the property: numbers $\varepsilon > 0$ and $\delta_0 > 0$ exists such that the condition

$$x[\eta]_\Delta \notin M^\varepsilon_{\eta, u_\eta[s]}$$

is fulfilled for every motion $\{x[\cdot]_\Delta, u[\cdot]\} = \{x[t, p_0, V^\circ]_\Delta, u[\cdot]\}$ with $\delta(\Delta) \leqslant \delta_0$, for any instant $\eta \in [t_0, \vartheta]$.

2. Let us indicate the conditions for the solvability of the problems posed. Let some set W be specified in the position space. We say that set W is (γ, u)-stable relative to M if for any $p_* = \{t_*, x_*, u_*(s)\} \in W, t^* \in (t_*, \vartheta], v(\cdot) \in Q([t_*, t^*))$ and $\gamma > 0$ we can find a function $u(\cdot) \in P([t_*, t^*))$ such that

$$x(t^*, p_*, u(\cdot), v(\cdot)) \in W^\gamma_{t^*, u_{t^*}(s)} \tag{2.1}$$

or if an instant $\eta \in [t_*, t^*]$ exists such that

$$x(\eta, p_*, u(\cdot), v(\cdot)) \in M^\gamma_{\eta, u_\eta(s)} \tag{2.2}$$

$$u_\eta(s) = \begin{cases} u(\eta + s), & s \in [t_* - \eta, 0) \\ u_*(\eta + s - t_*), & s \in [-\tau, t_* - \eta) \end{cases}$$

Here $x(t, p_*, u(\cdot), v(\cdot))$ is a solution of (1.1) from position p_* with the functions $u(\cdot)$ and $v(\cdot)$ selected (i.e., $x(t_*, p_*, u(\cdot), v(\cdot)) = x_*$ and $u(t_* + s) = u_*(s), -\tau \leqslant s < 0$). We say that set W is (γ, v)-stable if for any $p_* \in W, t^* \in (t_*, \vartheta], u(\cdot) \in P([t_*, t^*))$ and number $\gamma > 0$ we can find a function $v(\cdot) \in Q([t_*, t^*))$ such that condition (2.1) is fulfilled.

Let W be a set (γ, u)-stable relative to M, whose sections $W_{t, u(s)}$ are closed in R^n, i.e., $W = [W]$. A strategy U° associating with position $p_* = \{t_*, x_*, u_*(s)\}$ and number $t^* \in (t_*, \vartheta]$ a function $u^\circ(t) \in P([t_*, t^*))$ by the rule:

1°. Let $W_{t_*, u_*(s)} = \varnothing$. Then $u^\circ(t)$ is any function from $P([t_*, t^*))$;

2°. Let $W_{t_*, u_*(s)} \neq \varnothing$ and y be a vector from $W_{t_*, u_*(s)}$, closest to x_* in the metric of R^n, is called a strategy extremal to this set W. We choose the vector $v^* \in Q$ from the condition

$$(y - x_*) f_2(t_*, x_*, v^*) = \min_{v \in Q} \{(y - x_*) f_2(t_*, x_*, v)\} \tag{2.3}$$

Then we find a function $u^\circ(t) \in P([t_*, t^*))$ from the condition of (γ, u)-stability of set W with respect to the quantities $p_{**} = \{t_*, y, u_*(s)\} \in W$, t^*, function $v^*(t) \equiv v^*, t_* \leqslant t < t^*$ and number $\gamma \leqslant (t^* - t_*)^2$.

Let $W = [W]$. A strategy V° associating with position $p_* = \{t_*, x_*, u_*(s)\}$ and number $t^* \in (t_*, \vartheta]$ a function $v^\circ(t) \in Q([t_*, t^*))$ by the rule:

1°. Let $W_{t_*, u_*(s)} = \varnothing$. Then $v^\circ(t)$ is any function from $Q([t_*, t^*))$;

2°. Let $W_{t_*, u_*(s)} \neq \varnothing$ and vector $y \in W_{t_*, u_*(s)}$ be closest to x_* in R^n; is called a strategy extremal to W. Let vector $v^\circ \in Q$ satisfy the condition

$$(y - x_*) f_2(t_*, x_*, v^\circ) = \max_{v \in Q} \{(y - x_*) f_2(t_*, x_*, v)\}$$

Then $v^\circ(t) \equiv v^\circ,\ t \in [t_*, t^*)$.

On the space of positions $p_* = \{t_*, x_*, u_*(s)\}$ we introduce the function

$$r(p_*, W) = \begin{cases} \inf\{\|x_* - y\|\}, & W_{t_*, u_*(s)} \neq \varnothing \\ y \in W_{t_*, u_*(s)} & \\ +\infty, & W_{t_*, u_*(s)} = \varnothing \end{cases} \tag{2.4}$$

Lemma 2.1. Let set W be (γ, u)-stable relative to M and $W = [W]$. Then, if $p_0 \in W$, the strategy U° is extremal to W and ensures the condition: for any $\varepsilon > 0$ we can find $\delta_0 > 0$ such that for every motion $\{x[\cdot]_\Delta, u^\circ[\cdot]_\Delta\} = \{x[t, p_0, U^\circ]_\Delta, u^\circ[\cdot]_\Delta\}$ with $\delta(\Delta) \leqslant \delta_0$ the condition

$$r[\tau_i] = r(p_i, W) = r(\{\tau_i, x[\tau_i]_\Delta, u^\circ_{\tau_i}[s]_\Delta\}, W) \leqslant \varepsilon \tag{2.5}$$

is fulfilled for all $\tau_i \leqslant \tau_{i_1}$, where either τ_{i_1} is the instant τ_i at which the function $u^\circ[t] \in P([\tau_i, \tau_{i+1}))$ is first designated as the strategy U° from the condition (2.2) or $\tau_{i_1} = \vartheta$ if such an instant does not exist.

Let us sketch the lemma's proof. Let $X(t_0, x_0)$ be the set of solutions of system (1.1), corresponding to all possible functions $u(\cdot) \in P([t_0 - \tau, \vartheta])$, $v(\cdot) \in Q([t_0, \vartheta])$ with initial conditions $x(t_0) = x_0$. Let number $\lambda_1 = \lambda_1(t_0, x_0)$ be such that $\|x(t)\| \leqslant \lambda_1$, $t \in [t_0, \vartheta]$, $x(\cdot) \in X(t_0, x_0)$, and $\lambda_0 > 0$ be some number. We denote

$$X(t_0, x_0, \lambda_0) = \bigcup_{t_* \in [t_0, \vartheta],\ \|x_*\| \leqslant \lambda_1(t_0, x_0) + \lambda_0} X(t_*, x_*)$$

Then all the functions $x(\cdot) \in X(t_0, x_0, \lambda_0)$ are uniformly bounded by some constant $\lambda = \lambda(t_0, x_0, \lambda_0)$. Let us show that if the condition

$$r[\tau_i] \leqslant \lambda_0 = \varepsilon \tag{2.6}$$

is fulfilled for the motion $\{x[\cdot]_\Delta, u^\circ[\cdot]_\Delta\}$ and the instant $\tau_i < \tau_{i_1}$, then the estimate

$$r^2[\tau_{i+1}] \leqslant r^2[\tau_i](1 + C(\tau_{i+1} - \tau_i)) + (\tau_{i+1} - \tau_i)\varphi(\tau_{i+1} - \tau_i) \quad C = \text{const} \tag{2.7}$$

is valid. Here $\varphi(\delta)$ is a continuous function, $\varphi(\delta) \to 0$ as $\delta \to 0$, and the estimate (2.7) is uniform with respect to all motions $\{x[\cdot]_\Delta, u^\circ[\cdot]_\Delta\} = \{x[t, p_0, U^\circ]_\Delta, u^\circ[\cdot]_\Delta\}$ and instants τ_i with property (2.6), i.e., C and $\varphi(\delta)$ depend only on t_0, x_0 and λ_0. Indeed, by the choise of function $u^\circ[\cdot] \in P([\tau_i, \tau_{i+1}))$

$$r^2[\tau_{i+1}] \leqslant (\|x[\tau_{i+1}]_\Delta - z(\tau_{i+1})\| + \gamma)^2$$

where $z(t)$ is a solution of system (1.1) from the initial position $p_{*i} = \{\tau_i, y, u^\circ_{\tau_i}[s]_\Delta\} \in W$ with $v(t) \equiv v^*$, $t \in [\tau_i, \tau_{i+1})$ (v^* satisfies (2.3)), $y \in W_{\tau_i, u^\circ_{\tau_i}[s]_\Delta}$ is closest to $x[\tau_i]_\Delta$) and $u[t] \equiv u^\circ[t]_\Delta$. Since $\gamma \leqslant (\tau_{i+1} - \tau_i)^2$, by virtue of (2.6) we obtain

$$r^2[\tau_{i+1}] \leqslant \|x[\tau_{i+1}]_\Delta - z(\tau_{i+1})\|^2 + 2\lambda(\tau_{i+1} - \tau_i)^2 + (\tau_{i+1} - \tau_i)^4$$

Hence, because of the assumptions on the right-hand side of (1.1), we obtain

$$\begin{aligned} r^2[\tau_{i+1}] \leqslant \|x[\tau_i]_\Delta - y &+ \int_{\tau_i}^{\tau_{i+1}} f_1(t, x[t]_\Delta, u^\circ[t]_\Delta, u^\circ[t-\tau]_\Delta)\,dt + \\ &\int_{\tau_i}^{\tau_{i+1}} f_2(t, x[t]_\Delta, v[t])\,dt - \int_{\tau_i}^{\tau_{i+1}} f_1(t, z(t), u^\circ[t]_\Delta, u^\circ[t-\tau]_\Delta)\,dt - \\ &\int_{\tau_i}^{\tau_{i+1}} f_2(t, z(t), v^*)\,dt\,\|^2 + (2\lambda + (\tau_{i+1} - \tau_i)^2)(\tau_{i+1} - \tau_i)^2 \leqslant \\ &r^2[\tau_i] + 2\int_{\tau_i}^{\tau_{i+1}} (y - x[\tau_i]_\Delta)(f_2(t, x[\tau_i]_\Delta, v^*) - \\ &f_2(t, x[\tau_i]_\Delta, v[t]))\,dt + Cr^2[\tau_i] + (\tau_{i+1} - \tau_i)\varphi(\tau_{i+1} - \tau_i) \end{aligned}$$

By the choice of vector v^* we obtain estimate (2.7).

Now assume that the lemma is false. This signifies that we can find $\varepsilon > 0$ such that for any $\delta_0 > 0$, in particular, for δ_0 such that the estimate

$$(1 + \vartheta - t_0)\exp[c(\vartheta - t_0)]\varphi(\delta) \leqslant \varepsilon^2 \tag{2.8}$$

is fulfilled for $\delta < \delta_0$, we can find a motion $\{x[\cdot]_\Delta, u^\circ[\cdot]_\Delta\} = \{x[t, p_0, U^\circ]_\Delta, u^\circ[\cdot]_\Delta\}$ with $\delta(\Delta) \leqslant \delta_0$ and an instant $\tau_i \leqslant \tau_{i_1}$ such that (2.5) is not fulfilled. Let τ_{i_2} be the smallest partitioning instant τ_i at which condition (2.5) is not fulfilled. Then (2.6) is fulfilled for instants τ_i such that $\tau_0 \leqslant \tau_i < \tau_{i_2} < \tau_{i_1}$, which implies estimate (2.7). If the uniform estimate (2.7) is fulfilled for all $\tau_0 \leqslant \tau_i < \tau_{i_2}$, then the estimate

$$r^2[\tau_i] \leqslant (r^2[\tau_0] + (1 + \tau_i - t_0)\varphi(\delta)\exp[C(\tau_i - t_0)]$$

is fulfilled for all instants τ_i such that $\tau_0 \leqslant \tau_i \leqslant \tau_{i_2}$, which can be verified by contradiction. Hence by virtue of condition $r[\tau_0] = 0$ and of (2.8) follows $r[\tau_{i_2}] < \varepsilon$, which contradicts the definition of τ_{i_2}.

Theorem 2.1. Let set W be (γ, u)-stable relative to M, $W = [W]$ and $W_\vartheta \in [M_\vartheta]$. Then, if $p_0 \in W$, then the strategy U° extremal to W solves the problem of encounter with M.

Theorem 2.2. Let set W be (γ, v)-stable, $W = [W]$, and let $\varepsilon > 0$ exist such that $M^\varepsilon \cap W = \varnothing$. Then, if $p_0 \in W$, then the strategy V° extremal to W solves the problem of evading M.

Theorem 2.3. For any position p_0 and set M, either the problem of evading M is solvable or the problem of encounter with M^ε is solvable for any $\varepsilon > 0$.

The proofs of Theorems 2.1—2.3 rely on Lemma 2.1 and are analogous to the corresponding arguments in /1,3/.

Note. The following result is valid for systems without time lag in the control: if the problem of encounter with the target set is solvable for an initial position then in the position space there exists a stable set containing the initial position and terminating on the target set; therefore, the strategy resolving the encounter problem can be constructed as one extremal to the stable set /1,3/. In systems with aftereffect in the control this statement is, in general, false, as the following example shows. Consider the two-dimensional ($x = (x_1, x_2)$) system

$$x_1^{\cdot} = \begin{cases} 1 + (t - x_2(t))v(t), & t \in [-1, 0) \\ 1 - x_2(t)v(t), & t \in [0, 1] \end{cases}$$
$$x_2^{\cdot} = u(t - \tau)$$
$$t_0 = -1, \vartheta = 1, \tau = 1, |u| \leqslant 1, |v| \leqslant 1$$

Let the target set M consist of positions $p = \{t, x_1, x_2, u(s)\}$, where $t = \vartheta = 1, x_1 = 1, x_2 = 0, u(s)$ is any function from $P([-\tau, 0))$. By W we denote the set of positions from which the problem of encounter with M is solvable. The set $W_{-1} \neq \varnothing$ because the encounter problem is solvable from the position $p_0 = \{t_0, x_1^\circ, x_2^\circ, u_{t_0}(s)\}$, where $t_0 = -1$, $x_1^\circ = -1$, $x_2^\circ = -1$, $u_{t_0}(s) \equiv 1$, $s \in [-\tau, 0)$. However, $W_0 = \varnothing$ and, therefore, set W cannot be stable.

3. Let us show that the fundamental assertion in differential game theory, namely, the theorem on the alternative /1/, is valid for the differential encounter-evasion game made up of Problems 1 and 2.

In the position space let there be given a sequence of sets $\{W^{(j)}, j = 1, 2, \ldots\}$ with the properties:

1) $W^{(j+1)} \subset W^{(j)}$, 2) $W^{(j)} = [W^{(j)}]$, 3) set $W^{(j)}$ is (γ, u)-stable relative to M^{ε_j}, 4) $W_\vartheta^{(j)} \subset M_\vartheta^{\varepsilon_j}$, $\varepsilon_j = 1/j$. Let t_0, x_0 and number $\lambda_0 > 0$ also be given. On the set of positions $p_* = \{t_*, x_*, u_*(s)\}$ we introduce the function $\varkappa(p_*) = \varkappa(p_*, \{W^{(j)}\}, t_0, x_0, \lambda_0)$:

$$\varkappa(p_*) = \inf_j \{1/j \mid 1/j > r^2(p_*, W^{(j)})(1 + \vartheta - t_*)\exp \times [C(\vartheta - t_*)]\}$$

where function $r(p_*, W)$ is defined by (2.4) and $C = C(t_0, x_0, \lambda_0)$ is the constant in estimate (2.7). If the set

$$\{1/j \mid 1/j > r^2(p_*, W^{(j)})(1 + \vartheta - t_*)\exp[C(\vartheta - t_*)]\}$$

is empty, we assume $\varkappa(p_*) = +\infty$. We say that a strategy $U^{\circ\circ}$ is extremal to the sequence of sets $\{W^{(j)}\}$ with properties 1)—3) if /4,5/: $U^{\circ\circ}$ associates with position p_* and number $t^* \in (t_*, \vartheta]$ a function $u^{\circ\circ}(t) \in P([t_*, t^*))$ by the rule:

1°. Let $\varkappa(p_*) = +\infty$. Then $u^{\circ\circ}(t)$ is any function from $P([t_*, t^*))$.

2°. Let $\varkappa(p_*) < +\infty$. We find the number $j_0 = j_0(p_*)$ from the conditions: if $\varkappa(p_*) = 0$, then $1/j_0 < t^* - t_*$; if $0 < \varkappa(p_*) < +\infty$, then $1/j_0 = \varkappa(p_*)$. As $u^{\circ\circ}(\cdot)$ we take the function $u^\circ(t) \in P([t_*, t^*))$ designated as the strategy U° extremal to the (γ, u)-stable set $W^{(j_0)}$.

The following statements are valid.

Lemma 3.1. Let a sequence of sets $\{W^{(j)}\}$ with properties 1)—3), a position $p_0 = \{t_0, x_0, u_0(s)\} \in \bigcap_j W^{(j)}$ and $\lambda_0 \leqslant 1/4$ be specified. Then for any $\beta, 0 < \beta < \lambda_0$ we can find $\delta_0 > 0$ such that for every motion $\{x[\cdot]_\Delta, u^{\circ\circ}[\cdot]_\Delta\} = \{x[t, p_0, U^{\circ\circ}]_\Delta, u^{\circ\circ}[t]_\Delta\}$ with $\delta(\Delta) \leqslant \delta_0$ the condition $\varkappa[\tau_i] = \varkappa(p_i) = \varkappa(\{\tau_i, x[\tau_i]_\Delta, u_{\tau_i}{}^{\circ\circ}[s]_\Delta\}) < \beta$ is satisfied for all $\tau_i < \tau_{i_1}$, where either τ_{i_1} is the instant τ_i when the function $u^{\circ\circ}[t] \in P([\tau_i, \tau_{i+1})$, chosen as the function $u^\circ[t]$, is first fixed from condition (2.2) or $\tau_{i_1} = \vartheta$ if such an instant does not exist.

Lemma 3.2. In the hypotheses of Lemma 3.1 let the sequence of sets $\{W^{(j)}\}$ possess the property 4) as well. Then strategy $U^{\circ\circ}$ solves the problem of encounter with M for the position p_0.

From Theorem 2.3 and Lemma 3.2 follows

Theorem 3.1 (the alternative). For any position p_0 and any set M, either the problem of encounter with M or the problem of evading M is solvable.

Indeed, suppose that the problem of evading M is unsolvable from position p_0. Then by Theorem 2.3 the problem of encounter with $M^{\varepsilon,3}$ is solvable for any $\varepsilon/3 > 0$. This signifies that we can find a strategy U such that for the number $\varepsilon/3$ we can find a number $\delta_0 = \delta_0(\varepsilon/3, U) > 0$ such that for every motion $\{x[t]_\Delta, u[t]_\Delta\} = \{x[t, p_0, U]_\Delta, u[t]_\Delta\}$ with $\delta(\Delta) \leqslant \delta_0$ there exists $\eta \in [t_0, \vartheta]$ such that

$$x[\eta]_\Delta \in M^{2\varepsilon, 3}_{\eta, u_\eta[s]_\Delta}$$

Along such motions we compose the set $W(\varepsilon, U)$ of positions $\{t, x, u(s)\}$, $t \in [t_0, \eta]$, $x = x[t]_\Delta$, $u(s) = u_t[s]_\Delta$. The set $W(\varepsilon, U)$ is (γ, u)-stable relative to $M^{2\varepsilon, 3}$ and $W(\varepsilon, U)_\vartheta \subset M^{2\varepsilon, 3}_\vartheta$. But then the set $W(\varepsilon) = \bigcup W(\varepsilon, U)$, where the union is taken over all strategies U solving the problem of encounter with $M^{\varepsilon, 3}$ for p_0, is (γ, u)-stable relative to $M^{2\varepsilon, 3}$ and $W(\varepsilon)_\vartheta \subset M^{2\varepsilon, 3}_\vartheta$. We take the sequence $\varepsilon_j = 1/j$, $j = 1, 2, \ldots$; then the corresponding sequence of sets $\{W^{(j)} = [W(\varepsilon_j)], j = 1, 2, \ldots\}$ possesses properties 1)—4) and $p_0 \in \bigcap_j W^{(j)}$. Consequently, by Lemma 3.2, the problem of encounter with M is solvable. We note that, in general, the set $\bigcap_j W^{(j)}$ is not (γ, u)-stable (see the example).

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